Decompositions of Spin representations with respect to normalizers of spin subalgebras of low rank

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Abstract

We determine the decompositions of the Spin representations under the normalizer subalgebras of certain isomorphic copies of spin subalgebras $\mathfrak{spin}(r)$ within $\mathfrak{spin}(d_r m)$, where d_r is the dimension of a real irreducible representation of the even Clifford algebra Cl_r^0 of rank r (determined by the positive definite inner product on $\mathbb{R}^r$), and $r, m \in \mathbb{N}$.

1 Introduction

In this paper, we determine the decompositions of the Spin representations under the normalizer subalgebras $N_{\mathfrak{so}(d_r m)}(\mathfrak{spin}(r))$ of certain isomorphic copies of spin subalgebras $\mathfrak{spin}(r)$ within $\mathfrak{so}(d_r m)$, where d_r is the dimension of a real irreducible representation of the even Clifford algebra Cl_r^0 of rank r (determined by the positive definite inner product on $\mathbb{R}^r$), and $r, m \in \mathbb{N}$. The need to determine such decompositions arises in the geometries of Riemannian manifolds endowed with a even Clifford structures [7], and of Riemannian manifolds admitting spinorially twisted spin structures [**?**, 5].

Tha main idea is to rewrite the weights of the spin representations with respect to the ones of the normalizer subalgebras $N_{\mathfrak{so}(d_r m)}(\mathfrak{spin}(r))$.

The paper is organized as follows.

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2 Preliminaries

?⟨sec: prels⟩?

2.1 Representation theory of the classical Lie algebras

In this subsection we recall some results about the representation theory of the classical compact Lie groups and their Lie algebras [**?**].

2.1.1 Root system of $U(m)$

For $m \geq 2$, let $(\theta_1, \ldots, \theta_m)$ denote the coordinate vectors of elements in the maximal torus $\mathfrak{t}_{U(m)}$ of $U(m)$. The group $U(m)$ has the following root system:

- Roots: $\theta_\mu - \theta_\nu$, $\mu \neq \nu$, $1 \leq \mu, \nu \leq m$.
- Positive roots: $\theta_\mu - \theta_\nu$, $\mu < \nu$.
- Basis: $\theta_\nu - \theta_{\nu+1}$, $1 \leq \nu < m$.
- Fundamental Weyl chamber:

$$\bar{K} = \{(\theta_1, \ldots, \theta_m) \,:\, \theta_\nu \geq \theta_{\nu+1}, 1 \leq \nu < m\}.$$

- Sum of positive roots:

$$2\rho = \sum_{\nu=1}^{m} (m - 2\nu + 1)\theta_\nu.$$

2.1.2 Root system of $SU(m)$

Consider $SU(m) \subset U(m)$, the coordinates, root system and fundamental chamber described above.

- The Lie algebra $\mathfrak{su}(m)$ of $SU(m)$ is the subspace given by

$$\mathfrak{t}_{SU(m)} = \{(\theta_1, \ldots, \theta_m) \,:\, \theta_1 + \cdots + \theta_m = 0\}.$$

- The integral lattice is given by

$$I = \mathbb{Z} \cap \mathfrak{t}_{SU(m)} = \{(\theta_1, \ldots, \theta_m) \,:\, \theta_\nu \in \mathbb{Z}, \theta_1 + \cdots + \theta_m = 0\}$$

2.1.3 Root system of $SO(2m)$

For $m \geq 2$, the group $SO(2m)$ has the following root system:

- Roots: $\pm\theta_\mu \pm \theta_\nu$, $1 \leq \mu < \nu \leq m$.
- Positive roots: $\theta_\mu \pm \theta_\nu$, $1 \leq \mu < \nu \leq m$.
- Basis: $\theta_\nu - \theta_{\nu+1}$, $1 \leq \nu < m$, $\theta_{m-1} + \theta_m$.
- Fundamental Weyl chamber:

$$\bar{K} = \{(\theta_1, \ldots, \theta_m) \,:\, \theta_\nu \geq \theta_{\nu+1}, 1 \leq \nu < m, \theta_{\nu-1} \geq |\theta_m|\}.$$

- Sum of positive roots:

$$2\rho = \sum_{\nu=1}^{m} 2(m - \nu)\theta_\nu.$$

2.1.4 Root system of $SO(2m+1)$

For $m \geq 2$, the group $SO(2m+1)$ has the following root system:

- Roots: $\pm\theta_\mu \pm \theta_\nu$, $1 \leq \mu < \nu \leq m$ and $\pm\theta_\nu$, $1 \leq \nu \leq m$.
- Positive roots: $\theta_\mu \pm \theta_\nu$, $1 \leq \mu < \nu \leq m$, and θ_ν, $1 \leq \nu \leq m$
- Basis: $\theta_\nu - \theta_{\nu+1}$, $1 \leq \nu < m$, θ_m.
- Fundamental Weyl chamber:

$$\bar{K} = \{(\theta_1, \ldots, \theta_m) \,:\, \theta_\nu \geq \theta_{\nu+1}, 1 \leq \nu < m, \theta_n \geq 0\}.$$

- Sum of positive roots:

$$2\rho = \sum_{\nu=1}^{m} (2m - 2\nu + 1)\theta_\nu.$$

2.1.5 Root system of $Sp(m)$

For $m \geq 2$, the group $Sp(m)$ has the following root system:

- Roots: $\pm\theta_\mu \pm \theta_\nu$, $1 \leq \mu < \nu \leq m$, and $\pm 2\theta_m$, $1 \leq \nu \leq m$.
- Positive roots: $\theta_\mu \pm \theta_\nu$, $1 \leq \mu < \nu \leq m$, and $2\theta_m$, $1 \leq \nu \leq m$.
- Basis: $\theta_\nu - \theta_{\nu+1}$, $1 \leq \mu < \nu < m$, and $2\theta_m$.
- Fundamental Weyl chamber:

$$\bar{K} = \{(\theta_1, \ldots, \theta_m) \,:\, \theta_1 \geq \theta_2 \geq \cdots \geq \theta_m \geq 0\}.$$

- Sum of positive roots:

$$2\rho = \sum_{\nu=1}^{m} 2(m - \nu + 1)\theta_\nu.$$

2.1.6 Root system of $Spin(n)$

The root system of $Spin(n)$ is that of $SO(n)$. The integral latice of $Spin(n)$ is given by

$$I = \left\{ (\theta_1, \ldots, \theta_{[n/2]} \,:\, \theta_\nu \in \mathbb{Z}, 1 \leq \nu \leq [n/2], \sum_{\nu=1}^{[n/2]} \theta_\nu \in 2\mathbb{Z} \right\}.$$

the lattice of integral forma of $SPin(n)$ consists of linear forms

$$(\theta_1, \ldots, \theta_{[n/2]}) \mapsto x_1\theta_1 + \cdots + x_{[n/2]}\theta_{[n/2]}$$

such that either

$$x_\nu \in \mathbb{Z} \quad \text{for all } \nu,$$

or

$$x_\nu + \frac{1}{2} \in \mathbb{Z} \quad \text{for all } \nu.$$

2.2 Clifford algebra

⟨sec:preliminaries⟩ In this section we recall material that can also be consulted in [4, 6]. Let Cl_n denote the Clifford algebra generated by all the products of the orthonormal vectors $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ subject to the relations

$$e_j e_k + e_k e_j = -2\delta_{jk}, \quad \text{for } 1 \leq j,k \leq n.$$

We will often write

$$e_{1\ldots s} := e_1 e_2 \cdots e_s.$$

Let

$$\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C},$$

the complexification of Cl_n. It is well known that

$$\mathbb{C}l_n \cong \begin{cases} \mathrm{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k \\ \mathrm{End}(\mathbb{C}^{2^k}) \otimes \mathrm{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k+1 \end{cases},$$

where

$$\mathbb{C}^{2^k} = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$$

the tensor product of $k = [\frac{n}{2}]$ copies of $\mathbb{C}^2$. Let us denote

$$\Delta_n = \mathbb{C}^{2^k},$$

and consider the map

$$\kappa : \mathbb{C}l_n \longrightarrow \mathrm{End}(\mathbb{C}^{2^k})$$

which is an isomorphism for n even and the projection onto the first summand for n odd. In order to make κ_n explicit consider the following matrices with complex entries

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Now, consider the generators of the Clifford algebra $e_1, \ldots, e_n$ so that κ_n can be described as follows

$$\begin{array}{rcl} e_1 & \mapsto & Id \otimes Id \otimes \ldots \otimes Id \otimes Id \otimes g_1 \\ e_2 & \mapsto & Id \otimes Id \otimes \ldots \otimes Id \otimes Id \otimes g_2 \\ e_3 & \mapsto & Id \otimes Id \otimes \ldots \otimes Id \otimes g_1 \otimes T \\ e_4 & \mapsto & Id \otimes Id \otimes \ldots \otimes Id \otimes g_2 \otimes T \\ \vdots & & \ldots \\ e_{2k-1} & \mapsto & g_1 \otimes T \otimes \ldots \otimes T \otimes T \otimes T \\ e_{2k} & \mapsto & g_2 \otimes T \otimes \ldots \otimes T \otimes T \otimes T, \end{array}$$

and the last generator

$$e_{2k+1} \mapsto i\, T \otimes T \otimes \ldots \otimes T \otimes T \otimes T$$

if $n = 2k+1$.

Let

$$u_{+1} = \frac{1}{\sqrt{2}}(1, -i), \quad u_{-1} = \frac{1}{\sqrt{2}}(1, i)$$

which forms an orthonormal basis of $\mathbb{C}^2$ with respect to the standard Hermitian product. Note that

$$g_1(u_{\pm 1}) = i u_{\mp 1}, \quad g_2(u_{\pm 1}) = \pm u_{\mp 1}, \quad T(u_{\pm 1}) = \mp u_{\pm 1}.$$

Thus, we get a unitary basis of $\Delta_n = \mathbb{C}^{2^k}$

$$\mathcal{B} = \{u_{\varepsilon_1,\dots,\varepsilon_k} = u_{\varepsilon_1} \otimes \ldots \otimes u_{\varepsilon_k} \mid \varepsilon_j = \pm 1, j = 1, \dots, k\},$$

with respect to the induced Hermitian product on $\mathbb{C}^{2^k}$.

The Clifford multiplication of a vector e and a spinor ψ is defined by $e \cdot \psi = \kappa_n(e)(\psi)$. Thus, if $1 \leq j \leq k$

$$\begin{aligned} e_{2j-1} \cdot u_{\varepsilon_1,\dots,\varepsilon_k} &= i(-1)^{j-1} \left(\prod_{\alpha=k-j+2}^{k} \varepsilon_\alpha \right) u_{\varepsilon_1,\dots,(-\varepsilon_{k-j+1}),\dots,\varepsilon_k} \\ e_{2j} \cdot u_{\varepsilon_1,\dots,\varepsilon_k} &= (-1)^{j-1} \left(\prod_{\alpha=k-j+1}^{k} \varepsilon_\alpha \right) u_{\varepsilon_1,\dots,(-\varepsilon_{k-j+1}),\dots,\varepsilon_k} \end{aligned}$$

and

$$e_{2k+1} \cdot u_{\varepsilon_1,\dots,\varepsilon_k} = i(-1)^k \left(\prod_{\alpha=1}^{k} \varepsilon_\alpha \right) u_{\varepsilon_1,\dots,\varepsilon_k}$$

if $n = 2k+1$ is odd.

The Spin group $Spin(n) \subset Cl_n$ is the subset

$$Spin(n) = \{x_1 x_2 \cdots x_{2l-1} x_{2l} \mid x_j \in \mathbb{R}^n,\ |x_j| = 1,\ l \in \mathbb{N}\},$$

endowed with the product of the Clifford algebra. The Lie algebra of $Spin(n)$ is

$$\mathfrak{spin}(n) = \mathrm{span}\{e_i e_j \mid 1 \leq i < j \leq n\}.$$

The restriction of κ to $Spin(n)$ defines the Lie group representation

$$\kappa_n := \kappa|_{Spin(n)} : Spin(n) \longrightarrow GL(\Delta_n),$$

which is, in fact, special unitary [4].

There exist either real or quaternionic structures on the spin representations as described below [4]. A quaternionic structure α on $\mathbb{C}^2$ is given by

$$\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\overline{z}_2 \\ \overline{z}_1 \end{pmatrix},$$

and a real structure β on $\mathbb{C}^2$ is given by

$$\beta \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \overline{z}_1 \\ \overline{z}_2 \end{pmatrix}.$$

Note that these structures satisfy

$$\begin{aligned} \langle \alpha(v), w \rangle &= \overline{\langle v, \alpha(w) \rangle}, & \langle \alpha(v), \alpha(w) \rangle &= \overline{\langle v, w \rangle}, \\ \langle \beta(v), w \rangle &= \overline{\langle v, \beta(w) \rangle}, & \langle \beta(v), \beta(w) \rangle &= \overline{\langle v, w \rangle}, \end{aligned}$$

with respect to the standard hermitian product in $\mathbb{C}^2$, where $v, w \in \mathbb{C}^2$. The real and quaternionic structures γ_n on $\Delta_n = (\mathbb{C}^2)^{\otimes [n/2]}$ are built as follows

$$\begin{aligned} \gamma_n &= (\alpha \otimes \beta)^{\otimes 2k} && \text{if } n = 8k, 8k+1 && \text{(real)}, \\ \gamma_n &= \alpha \otimes (\beta \otimes \alpha)^{\otimes 2k} && \text{if } n = 8k+2, 8k+3 && \text{(quaternionic)}, \\ \gamma_n &= (\alpha \otimes \beta)^{\otimes 2k+1} && \text{if } n = 8k+4, 8k+5 && \text{(quaternionic)}, \\ \gamma_n &= \alpha \otimes (\beta \otimes \alpha)^{\otimes 2k+1} && \text{if } n = 8k+6, 8k+7 && \text{(real)}. \end{aligned}$$

which also satisfy

$$\langle \gamma_n(v), w\rangle = \overline{\langle v, \gamma_n(w)\rangle}, \qquad \langle \gamma_n(v), \gamma_n(w)\rangle = \overline{\langle v, w\rangle},$$

where $v, w \in \Delta_n$. This means

$$\langle v + \gamma_n(v), w + \gamma_n(w)\rangle \in \mathbb{R}. \tag{1}$$

?eq: real inner produ

The Spin group $Spin(n) \subset Cl_n$ is the subset

$$Spin(n) = \{x_1 x_2 \cdots x_{2l-1} x_{2l} \mid x_j \in \mathbb{R}^n,\ |x_j| = 1,\ l \in \mathbb{N}\},$$

endowed with the product of the Clifford algebra. The Lie algebra of $Spin(n)$ is

$$\mathfrak{spin}(n) = \mathrm{span}\{e_i e_j \mid 1 \le i < j \le n\}.$$

The restriction of κ to $Spin(n)$ defines the Lie group representation

$$\kappa_n := \kappa|_{Spin(n)} : Spin(n) \longrightarrow GL(\Delta_n),$$

which is, in fact, special unitary [4].

2.2.1 Weights of the Spin representation

Recall that a maximal torus of $SO(n)$ is given by

$$\begin{pmatrix} \cos(\eta_1) & -\sin(\eta_1) & & & & & \\ \sin(\eta_1) & \cos(\eta_1) & & & & & \\ & & \cos(\eta_2) & -\sin(\eta_2) & & & \\ & & \sin(\eta_2) & \cos(\eta_2) & & & \\ & & & & \ddots & & \\ & & & & & \cos(\eta_{n/2}) & -\sin(\eta_{n/2}) \\ & & & & & \sin(\eta_{n/2}) & \cos(\eta_{n/2}) \end{pmatrix} \tag{2}$$

eq: torus SO(n) n eve

if n is even, and

$$\begin{pmatrix} \cos(\eta_1) & -\sin(\eta_1) & & & & & & \\ \sin(\eta_1) & \cos(\eta_1) & & & & & & \\ & & \cos(\eta_2) & -\sin(\eta_2) & & & & \\ & & \sin(\eta_2) & \cos(\eta_2) & & & & \\ & & & & \ddots & & & \\ & & & & & \cos(\eta_{(n-1)/2]}) & -\sin(\eta_{(n-1)/2}) & \\ & & & & & \sin(\eta_{(n-1)/2}) & \cos(\eta_{(n-1)/2}) & \\ & & & & & & & 1 \end{pmatrix} \tag{3}$$

eq: torus SO(n) n odd

if n is odd. Each one of those 2×2 blocks is a rotation in a canonical 2-plane, which is an transformation that can be achieved by using Clifford product. For instance the rotation

$$\begin{pmatrix} \cos(\eta_1) & -\sin(\eta_1) & & & \\ \sin(\eta_1) & \cos(\eta_1) & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

can be achieved by using the element

$$e_1(-\cos(\eta_1/2)e_1 + \sin(\eta_1/2)e_2) = \cos(\eta_1/2) + \sin(\eta_1/2)e_1e_2 \in Spin(n)$$

as follows

$$\begin{aligned}
&(\cos(\eta_1/2) + \sin(\eta_1/2)e_1e_2)y(\cos(\eta_1/2) + \sin(\eta_1/2)e_1e_2) \\
&= (y_1\cos(\eta_1) - y_2\sin(\eta_1))e_1 + (y_1\sin(\eta_1) + y_2\cos(\eta_1)e_2 + y_3e_3 + \cdots + y_ne_n,
\end{aligned}$$

for $y = y_1e_1 + \cdots + y_ne_n \in \mathbb{R}^n$. Thus we see that the corresponding element in $Spin(n)$ is exactly

$$\cos(\eta_1/2) + \sin(\eta_1/2)e_1e_2.$$

Furthermore, we can see that a maximal torus of $Spin(n)$ is given by the elements

$$t(\eta_1, \dots, \eta_k) = \prod_{j=1}^{k}(\cos(\eta_j/2) + \sin(\eta_j/2)e_{2j-1}e_{2j}),$$

where $k = [n/2]$. By using the explicit description of the isomorphisms given above, we can check that

$$t(\eta_1, \dots, \eta_k) \cdot u_{\varepsilon_1,\dots\varepsilon_k} = e^{\frac{i}{2}\sum_{j=1}^{k}\varepsilon_{k-j}\eta_j} \cdot u_{\varepsilon_1,\dots\varepsilon_k},$$

i.e. the basis vectors $u_{\varepsilon_1,\dots\varepsilon_k}$ are weight vectors of the spin representation with weight

$$\frac{1}{2}\sum_{j=1}^{k}\varepsilon_{k-j}\eta_j,$$

which in coordinate vectors with respect to the basis $\{\eta_j\}$ are the well known expressions

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \cdots, \pm\frac{1}{2}\right).$$

It will be these weights that will be rewritten with respect to the coordinates of the tori of the normalizers of spin subalgebras.

2.3 Clifford and Spin representations

Now, we summarize some results about real representations of Cl_r^0 in the next table (cf. [6]). Here d_r denotes the dimension of an irreducible representation of Cl_r^0 and v_r the number of distinct irreducible representations.

$r \pmod 8$	Cl_r^0	d_r	v_r
1	$\mathbb{R}(d_r)$	$2^{\lfloor \frac{r}{2} \rfloor}$	1
2	$\mathbb{C}(d_r/2)$	$2^{\frac{r}{2}}$	1
3	$\mathbb{H}(d_r/4)$	$2^{\lfloor \frac{r}{2} \rfloor + 1}$	1
4	$\mathbb{H}(d_r/4) \oplus \mathbb{H}(d_r/4)$	$2^{\frac{r}{2}}$	2
5	$\mathbb{H}(d_r/4)$	$2^{\lfloor \frac{r}{2} \rfloor + 1}$	1
6	$\mathbb{C}(d_r/2)$	$2^{\frac{r}{2}}$	1
7	$\mathbb{R}(d_r)$	$2^{\lfloor \frac{r}{2} \rfloor}$	1
8	$\mathbb{R}(d_r) \oplus \mathbb{R}(d_r)$	$2^{\frac{r}{2}-1}$	2

Table 1

Let $\tilde{\Delta}_r$ denote the irreducible representation of Cl_r^0 for $r \not\equiv 0 \pmod 4$ and $\tilde{\Delta}_r^{\pm}$ denote the irreducible representations for $r \equiv 0 \pmod 4$. Note that the representations are complex for $r \equiv 2, 6 \pmod 8$ and quaternionic for $r \equiv 3, 4, 5 \pmod 8$. It is interesting to note that these features are reflected in the main results of the paper.

Note also that if $r \equiv 4, 6, 7, 8 \pmod 8$ then $d_r = d_{r-1}$ and if $r \equiv 1, 2, 3, 5 \pmod 8$ then $d_r = 2d_{r-1}$. By restricting to a standard subagebra $Cl_{r-1}^0 \subset Cl_r^0$, the representations decompose as follows:

$r \pmod 8$	$\tilde{\Delta}_r\vert_{Cl_{r-1}^0}$
1	$\tilde{\Delta}_r \cong \tilde{\Delta}_{r-1}^{+} + \tilde{\Delta}_{r-1}^{-}$
2	$\tilde{\Delta}_r \cong \tilde{\Delta}_{r-1} + \tilde{\Delta}_{r-1}$
3	$\tilde{\Delta}_r \cong \tilde{\Delta}_{r-1} + \tilde{\Delta}_{r-1}$
4	$\tilde{\Delta}_r^{\pm} \cong \tilde{\Delta}_{r-1}$
5	$\tilde{\Delta}_r \cong \tilde{\Delta}_{r-1}^{+} + \tilde{\Delta}_{r-1}^{-}$
6	$\tilde{\Delta}_r \cong \tilde{\Delta}_{r-1}$
7	$\tilde{\Delta}_r \cong \tilde{\Delta}_{r-1}$
8	$\tilde{\Delta}_r^{\pm} \cong \tilde{\Delta}_{r-1}$

Table 2

2.4 Normalizers

?⟨sec: normalizers⟩?

Due to geometric considerations in [7, 5], we will consider $\mathfrak{spin}(r)$ embedded in $\mathfrak{so}(N)$ in the following way. Suppose that Cl_r^0 is represented on $\mathbb{R}^N$, for some $N \in \mathbb{N}$, in such a way that each bivector $e_i e_j$ is mapped to an antisymmetric endomorphism J_{ij} satisfying

$$J_{ij}^2 = -\mathrm{Id}_{\mathbb{R}^N}. \tag{4}$$

eq:almost-complex-str

Normalizer of $\mathfrak{spin}(r)$ in $\mathfrak{so}(d_r m)$, $r \not\equiv 0 \pmod 4$, $r > 1$

entralizer r not 4⟩?

Let us assume $r \not\equiv 0 \pmod 4$, $r > 1$. In this case, $\mathbb{R}^N$ decomposes into a sum of irreducible representations of Cl_r^0. Since this algebra is simple, such irreducible representations can only be trivial or copies of the standard representation $\tilde{\Delta}_r$ of Cl_r^0 (cf. [6]). Due to (4), there are no trivial summands in such a decomposition so that

$$\mathbb{R}^N = \underbrace{\tilde{\Delta}_r \oplus \cdots \oplus \tilde{\Delta}_r}_{m\ times},$$

for some $m \in \mathbb{N}$. By restricting to $\mathfrak{spin}(r) \subset Cl_r^0$,

$$\mathbb{R}^N = \tilde{\Delta}_r \otimes_{\mathbb{R}} \mathbb{R}^m,$$

we see that $N = d_r m$ and $\mathfrak{spin}(r)$ has an isomorphic image

$$\widehat{\mathfrak{spin}(r)} = \mathfrak{spin}(r) \otimes \{\mathrm{Id}_{m\times m}\} \subset \mathfrak{so}(d_r m),$$

which is the subalgebra of $\mathfrak{so}(d_r m)$ whose centralizer $C_{\mathfrak{so}(d_r m)}(\widehat{\mathfrak{spin}(r)})$ and normalizer $N_{\mathfrak{so}(d_r m)}(\widehat{\mathfrak{spin}(r)})$ are given in the following theorem.

?⟨theo: 1⟩?

Theorem 2.1 [1] *Let* $r \not\equiv 0 \pmod 4$ *and let* $\widehat{\mathfrak{spin}(r)} \subset \mathfrak{so}(d_r m)$ *as described before. The centralizer and normalizer of* $\widehat{\mathfrak{spin}(r)}$ *in* $\mathfrak{so}(d_r m)$ *are isomorphic to*

$r \pmod 8$	$C_{\mathfrak{so}(d_r m)}(\widehat{\mathfrak{spin}(r)})$	$N_{\mathfrak{so}(d_r m)}(\widehat{\mathfrak{spin}(r)})$
± 1	$\mathfrak{so}(m)$	$\mathfrak{so}(m) \oplus \mathfrak{spin}(r)$
± 2	$\mathfrak{u}(m)$	$\mathfrak{u}(m) \oplus \mathfrak{spin}(r)$
± 3	$\mathfrak{sp}(m)$	$\mathfrak{sp}(m) \oplus \mathfrak{spin}(r)$

Normalizer of $\mathfrak{spin}(r)$ in $\mathfrak{so}(d_r m_1 + d_r m_2)$, $r \equiv 0 \pmod 4$

ntralizer-Spin(4k)⟩?
Let us assume $r \equiv 0 \pmod 4$. Recall that if $\hat{\Delta}_r$ is the irreducible representation of Cl_r, then by restricting this representation to Cl_r^0 it splits as the sum of two inequivalent irreducible representations

$$\hat{\Delta}_r = \tilde{\Delta}_r^+ \oplus \tilde{\Delta}_r^-.$$

Since $\mathbb{R}^N$ is a representation of Cl_r^0 satisfying (4), there are no trivial summands in such a decomposition so that

$$\mathbb{R}^N \quad = \quad \tilde{\Delta}_r^+ \otimes \mathbb{R}^{m_1} \oplus \tilde{\Delta}_r^- \otimes \mathbb{R}^{m_2},$$

for some $m_1, m_2 \in \mathbb{N}$. By restricting this representation to $\mathfrak{spin}(r) \subset Cl_r^0$, consider

$$\widehat{\mathfrak{spin}(r)} = \mathfrak{spin}(r)^+ \otimes (\mathrm{Id}_{m_1 \times m_1} \oplus \mathbf{0}_{m_2 \times m_2}) \oplus \mathfrak{spin}(r)^- \otimes (\mathbf{0}_{m_1 \times m_1} \oplus \mathrm{Id}_{m_2 \times m_2}) \subset \mathfrak{so}(d_r m_1 + d_r m_2),$$

where $\mathfrak{spin}(r)^{\pm}$ are the images of $\mathfrak{spin}(r)$ in $\mathrm{End}(\tilde{\Delta}_r^{\pm})$ respectively.

?⟨theo: 2⟩?
Theorem 2.2 [1] *Let* $r \equiv 0 \pmod 4$. *The centralizer and normalizer of* $\widehat{\mathfrak{spin}(r)}$ *in* $\mathfrak{so}(d_r m_1 + d_r m_2)$ *are isomorphic to*

$r \pmod 8$	$C_{\mathfrak{so}(d_r m_1 + d_r m_2)}(\widehat{\mathfrak{spin}(r)})$	$N_{\mathfrak{so}(d_r m_1 + d_r m_2)}(\widehat{\mathfrak{spin}(r)})$
0	$\mathfrak{so}(m_1) \oplus \mathfrak{so}(m_2)$	$\mathfrak{so}(m_1) \oplus \mathfrak{so}(m_2) \oplus \mathfrak{spin}(r)$
4	$\mathfrak{sp}(m_1) \oplus \mathfrak{sp}(m_2)$	$\mathfrak{sp}(m_1) \oplus \mathfrak{sp}(m_2) \oplus \mathfrak{spin}(r)$

3 Decompositions

3.1 $r = 2$

In this case, $Spin(2)$ is the connected double cover of $SO(2) \cong U(1)$, and is therefore isomorphic to $U(1)$. In particular, $\mathfrak{spin}(2) \cong \mathfrak{u}(1)$ is contained in its centralizer, i.e.

$$N_{\mathfrak{so}(2m)}(\mathfrak{spin}(2)) = \mathfrak{u}(m).$$

Recall that $\mathbb{C}^m \cong \mathbb{R}^{2m}$ by considering the underlying real vector space and that

$$U(m) \hookrightarrow SO(2m).$$

Furthermore,

$$\mathbb{R}^{2m} \otimes \mathbb{C} = \mathbb{C}^m \oplus \overline{\mathbb{C}^m}.$$

In terms of the standard represntation of $SO(2m)$, its standard maximal torus looks like (2) or (3). In terms of the complexified standard representation $\mathbb{R}^{2m} \otimes \mathbb{C}$ of $SO(2m)$, its standard maximal torus looks like

$$\begin{pmatrix} e^{i\eta_1} & & & & & & \\ & e^{-i\eta_1} & & & & & \\ & & e^{i\eta_2} & & & & \\ & & & e^{-i\eta_2} & & & \\ & & & & \ddots & & \\ & & & & & e^{i\eta_m} & \\ & & & & & & e^{-i\eta_m} \end{pmatrix}$$

The standard maximal torus of $U(m)$ looks

$$\begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_m} \end{pmatrix}$$

which represented on

$$\mathbb{R}^{2m} \otimes \mathbb{C} = \mathbb{C}^m \oplus \overline{\mathbb{C}^m}$$

with an appropriately ordered basis looks like

$$\begin{pmatrix} e^{i\theta_1} & & & & \\ & e^{-i\theta_1} & & & \\ & & \ddots & & \\ & & & e^{i\theta_m} & \\ & & & & e^{-i\theta_m} \end{pmatrix},$$

so that we can identify

$$\eta_j = \theta_j$$

for $j = 1, \ldots, m$.

On the other hand, the representation $\mathbb{C}^m$ of $U(m)$ factors as

$$\mathbb{C}^m \cong L \otimes E$$

with respect to the isomorphism

$$U(m) \cong U(1) \times_{\mathbb{Z}_m} SU(m),$$

where L and E denote the fundamental 1-dimensional and m-dimensional representations of $U(1)$ and $SU(m)$ respectively. The standard maximal torus of $U(m)$ factors accordingly as

$$\begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_m} \end{pmatrix} = e^{i\theta'} \begin{pmatrix} e^{i\theta_1''} & & & \\ & e^{i\theta_2''} & & \\ & & \ddots & \\ & & & e^{-i(\theta_1''+\cdots+\theta_{m-1}'')} \end{pmatrix}$$

so that

$$\eta_j = \theta_j = \theta' + \theta_j''$$

for $j = 1, \ldots, m-1$, and

$$\eta_m = \theta_m = \theta' - (\theta_1'' + \cdots + \theta_{m-1}'')$$

Note that each weight

$$\pm\frac{\eta_1}{2} \pm \cdots \pm \frac{\eta_{m-1}}{2} + \frac{\eta_m}{2}$$

of the spin representation can be written as

$$\pm\frac{\eta_1}{2} \pm \cdots \pm \frac{\eta_{m-1}}{2} + \frac{\eta_m}{2} = \sum_{j\in I}\frac{\eta_j}{2} - \sum_{j\in \bar{I}}\frac{\eta_j}{2} + \frac{\eta_m}{2},$$

where $I \subset \{1, \ldots, m-1\}$ and $\bar{I} = \{1, \ldots, m-1\} - I$. By substituting we get

$$\begin{aligned}
\sum_{j\in I}\frac{\eta_j}{2} - \sum_{j\in \bar{I}}\frac{\eta_j}{2} + \frac{\eta_m}{2} &= \sum_{j\in I}\frac{\theta'+\theta_j''}{2} - \sum_{j\in \bar{I}}\frac{\theta'+\theta_j''}{2} + \frac{\theta' - (\theta_1''+\cdots+\theta_{m-1}'')}{2} \\
&= \frac{(|I| - |\bar{I}| + 1)}{2}\theta' + \sum_{j\in I}\frac{\theta_j''}{2} - \sum_{j\in \bar{I}}\frac{\theta_j''}{2} - \left(\sum_{j\in I}\frac{\theta_j''}{2} + \sum_{j\in \bar{I}}\frac{\theta_j''}{2}\right) \\
&= \frac{(|I| - (m-1-|I|) + 1)}{2}\theta' + \left(\sum_{j\in I}\frac{\theta_j''}{2} - \sum_{j\in I}\frac{\theta_j''}{2}\right) - \left(\sum_{j\in \bar{I}}\frac{\theta_j''}{2} + \sum_{j\in \bar{I}}\frac{\theta_j''}{2}\right) \\
&= \frac{(2|I| - m + 2)}{2}\theta' - \sum_{j\in \bar{I}}\theta_j''.
\end{aligned}$$

Similarly for the weights

$$\pm\frac{\eta_1}{2} \pm \cdots \pm \frac{\eta_{m-1}}{2} - \frac{\eta_m}{2}$$

we get

$$\begin{aligned}
\sum_{j\in I}\frac{\eta_j}{2} - \sum_{j\in \bar{I}}\frac{\eta_j}{2} - \frac{\eta_m}{2} &= \sum_{j\in I}\frac{\theta'+\theta_j''}{2} - \sum_{j\in \bar{I}}\frac{\theta'+\theta_j''}{2} - \frac{\theta' - (\theta_1''+\cdots+\theta_{m-1}'')}{2} \\
&= \frac{(|I| - |\bar{I}| - 1)}{2}\theta' + \sum_{j\in I}\frac{\theta_j''}{2} - \sum_{j\in \bar{I}}\frac{\theta_j''}{2} + \left(\sum_{j\in I}\frac{\theta_j''}{2} + \sum_{j\in \bar{I}}\frac{\theta_j''}{2}\right) \\
&= \frac{(|I| - (m-1-|I|) - 1)}{2}\theta' + \left(\sum_{j\in I}\frac{\theta_j''}{2} + \sum_{j\in I}\frac{\theta_j''}{2}\right) - \left(\sum_{j\in \bar{I}}\frac{\theta_j''}{2} - \sum_{j\in \bar{I}}\frac{\theta_j''}{2}\right)
\end{aligned}$$

$$= \frac{(2|I|-m)}{2}\theta' + \sum_{j\in I}\theta''_j.$$

From here we recognize:

- the $U(1)$ representation needs to have a square root if m is odd;
- the weights $\frac{(2|I|-m)}{2}\theta'$ correspond to tensor powers of the line $L^{1/2}$;
- the weights $\pm\sum_{j\in I}\theta''_j$ correspond to pieces of the characters of exterior powers of E.

Thus, we have recognized that

$$\begin{aligned}\Delta_{2m} &\cong \bigoplus_{j=0}^{m}(L^{1/2})^{\otimes m-2j}\otimes \bigwedge^{j}E \\ &\cong \bigoplus_{j=0}^{m}(L^{1/2})^{\otimes m-2j}\otimes V^{\mathfrak{su}(m)}(\underbrace{1,\ldots,1}_{j},\underbrace{0,\ldots,0}_{m-1-j}),\end{aligned}$$

which is consistent with the description given in in [8, Sectioon 3]. Note that the numbers

$$[m/2-j]+1\times j+0\times(m-1-j)=m/2.$$

3.2 $r = 3$

This case was developed in [3]. Recall that $\mathfrak{spin}(3) \cong \mathfrak{sp}(1)$ and

$$N_{\mathfrak{so}(4m)}(\mathfrak{spin}(3)) = \mathfrak{sp}(m) \oplus \mathfrak{sp}(1).$$

In this case

$$\mathbb{R}^{4m} \otimes \mathbb{C} \cong E \otimes H$$

where $E = \mathbb{C}^{2m}$ and $H = \mathbb{C}^2$ are the standard representations of $\mathfrak{sp}(m)$ and $\mathfrak{sp}(1)$ respectively.

In terms of the complexified standard representation $\mathbb{R}^{4m} \otimes \mathbb{C}$ of $SO(4m)$, its standard maximal torus looks like

$$\begin{pmatrix} e^{i\eta_1} & & & & & & \\ & e^{-i\eta_1} & & & & & \\ & & e^{i\eta_2} & & & & \\ & & & e^{-i\eta_2} & & & \\ & & & & \ddots & & \\ & & & & & e^{i\eta_{2m}} & \\ & & & & & & e^{-i\eta_{2m}} \end{pmatrix}$$

The standard maximal torus of $Sp(m)$ looks

$$\begin{pmatrix} e^{i\theta_1} & & & & \\ & e^{-i\theta_1} & & & \\ & & \ddots & & \\ & & & e^{i\theta_m} & \\ & & & & e^{-i\theta_m} \end{pmatrix}$$

and the standard maximal torus of $Sp(1)$ looks like

$$\begin{pmatrix} e^{i\varphi} & \\ & e^{-i\varphi} \end{pmatrix}.$$

Clearly, we must take the Kronecker product of these two matrices so that we have the correspondence

$$\begin{aligned} \eta_{2j-1} &= \theta_j + \varphi \\ \eta_{2j} &= \theta_j - \varphi. \end{aligned}$$

Now we substitute this identities into the weights of the Spin representation

$$\begin{aligned} \pm\frac{\eta_1}{2} \pm \cdots \pm \frac{\eta_{2m}}{2} &= \sum_{j\in I_1} \frac{\theta_j + \varphi}{2} - \sum_{j\in \bar{I}_1} \frac{\theta_j + \varphi}{2} + \sum_{j\in I_2} \frac{\theta_j - \varphi}{2} - \sum_{j\in \bar{I}_2} \frac{\theta_j - \varphi}{2} \\ &= \frac{(|I_1| - |\bar{I}_1| - |I_2| + |\bar{I}_2|)}{2}\varphi + \frac{1}{2}\left(\sum_{j\in I_1}\theta_j - \sum_{j\in \bar{I}_1}\theta_j + \sum_{j\in I_2}\theta_j - \sum_{j\in \bar{I}_2}\theta_j\right) \\ &= \frac{(|I_1| - (m - |I_1|) - |I_2| + (m - |I_2|))}{2}\varphi \\ &\quad + \frac{1}{2}\left(\sum_{j\in I_1\cap I_2} + \theta_j \sum_{j\in I_1 - I_2}\theta_j - \sum_{j\in \bar{I}_1\cap \bar{I}_2}\theta_j - \sum_{j\in \bar{I}_1 - \bar{I}_2}\theta_j \right. \\ &\quad \left. + \sum_{j\in I_2\cap I_1}\theta_j + \sum_{j\in I_2 - I_1}\theta_j - \sum_{j\in \bar{I}_1\cap \bar{I}_2}\theta_j - \sum_{j\in \bar{I}_2 - \bar{I}_1}\theta_j\right) \end{aligned}$$

$$
\begin{aligned}
&= (|I_1|-|I_2|)\varphi \\
&\quad +\frac{1}{2}\left(2\sum_{j\in I_1\cap I_2}\theta_j - 2\sum_{j\in \bar{I}_1\cap \bar{I}_2}\theta_j + \sum_{j\in I_1-I_2}\theta_j - \sum_{j\in \bar{I}_1-\bar{I}_2}\theta_j + \sum_{j\in I_2-I_1}\theta_j - \sum_{j\in \bar{I}_2-\bar{I}_1}\theta_j\right) \\
&= (|I_1|-|I_2|)\varphi + \sum_{j\in I_1\cap I_2}\theta_j - \sum_{j\in \bar{I}_1\cap \bar{I}_2}\theta_j,
\end{aligned}
$$

where $I_1, I_2 \subseteq \{1,\ldots,m\}$ and

$$
\begin{aligned}
I_1 - I_2 &= \bar{I}_2 - \bar{I}_1, \\
I_2 - I_1 &= \bar{I}_1 - \bar{I}_2.
\end{aligned}
$$

From here we recognize the highest weights of $\mathfrak{sp}(1)$ and $\mathfrak{sp}(n)$ so that [3]

$$
\begin{aligned}
\Delta_{4m} &= \bigoplus_{j=0}^{m} \textstyle\bigwedge_0^{m-j} E \otimes S^j H. \\
&= \bigoplus_{j=0}^{m} V^{\mathfrak{sp}(m)}(\underbrace{1,\ldots,1}_{m-j},\underbrace{0,\ldots,0}_{j}) \otimes V^{\mathfrak{sp}(1)}(j).
\end{aligned}
$$

Note that the numbers

$$
1\times(m-j)+0\times j+[j]=m,
$$

i.e. they add up to m.

We have a linear map

$$
\mathfrak{t}_{sp(m)} \oplus \mathfrak{t}_{sp(1)} \hookrightarrow \mathfrak{t}_{SO(4m)}
$$

expressed as

$$
\begin{pmatrix}
1 & & & & & 1\\
1 & & & & & -1\\
 & 1 & & & & 1\\
 & 1 & & & & -1\\
 & & 1 & & & 1\\
 & & 1 & & & -1\\
 & & & 1 & & 1\\
 & & & 1 & & -1\\
 & & & & 1 & 1\\
 & & & & 1 & -1
\end{pmatrix}
\begin{pmatrix}
x[1]\\ x[2]\\ x[3]\\ x[4]\\ x[5]\\ y
\end{pmatrix}
=
\begin{pmatrix}
x[1]+y\\ x[1]-y\\ x[2]+y\\ x[2]-y\\ x[3]+y\\ x[3]-y\\ x[4]+y\\ x[4]-y\\ x[5]+y\\ x[5]-y
\end{pmatrix}
$$

where we have set $m=5$ for the sake of clarity.

The transpose is

$$
\begin{pmatrix}
1 & 1 & & & & & & & & \\
 & & 1 & 1 & & & & & & \\
 & & & & 1 & 1 & & & & \\
 & & & & & & 1 & 1 & & \\
 & & & & & & & & 1 & 1\\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
\eta[1]\\ \eta[2]\\ \eta[3]\\ \eta[4]\\ \eta[5]\\ \eta[6]\\ \eta[7]\\ \eta[8]\\ \eta[9]\\ \eta[10]
\end{pmatrix}
=
\begin{pmatrix}
\eta[1]+\eta[2]\\ \eta[3]+\eta[4]\\ \eta[5]+\eta[6]\\ \eta[7]+\eta[8]\\ \eta[9]+\eta[10]
\end{pmatrix}
$$

3.3 $r = 4$

Recall that $\mathfrak{spin}(4) \cong \mathfrak{sp}(1)_1 \oplus \mathfrak{sp}(1)_2$ and

$$N_{\mathfrak{so}(4m)}(\mathfrak{spin}(4)) = \mathfrak{sp}(m_1) \oplus \mathfrak{sp}(m_2) \oplus \mathfrak{spin}(4).$$

for some $m_1, m_2 \in \mathbb{N}$ such that $m_1 + m_2 = m$. In this case

$$\mathbb{R}^{4m} = \mathbb{R}^{m_1} \otimes \tilde{\Delta}_4^+ \oplus \mathbb{R}^{m_2} \otimes \tilde{\Delta}_4^-$$

and

$$\mathbb{R}^{4m} \otimes \mathbb{C} \cong E_1 \otimes H_1 \oplus E_2 \otimes H_2$$

where $E_j = \mathbb{C}^{2m}$ and $H_j = \mathbb{C}^2$ are the standard representations of $\mathfrak{sp}(m_j)$ and $\mathfrak{sp}(1)_j$ respectively. From the explicit description of the spinor representation we can see that

$$\Delta_{4m} \cong \Delta_{4m_2} \otimes \Delta_{4m_1}$$

so that the restriction to $N_{\mathfrak{so}(4m)}(\mathfrak{spin}(4))$ factors as follows

$$\begin{aligned}
\Delta_{4m} &= \left(\bigoplus_{j=0}^{m_2} \textstyle\bigwedge_0^{m_2-j} E_2 \otimes S^j H_2\right) \otimes \left(\bigoplus_{k=0}^{m_1} \textstyle\bigwedge_0^{m_1-k} E_1 \otimes S^k H_1\right) \\
&= \left(\bigoplus_{j=0}^{m_2} \textstyle\bigwedge_0^{m_2-j} E_2 \otimes S^j H_2\right) \otimes \left(\bigoplus_{k=0}^{m_1} \textstyle\bigwedge_0^{m_1-k} E_1 \otimes S^k H_1\right). \\
&= \left(\bigoplus_{j=0}^{m_2} V^{\mathfrak{sp}(m_2)}(\underbrace{1,\ldots,1}_{m_2-j},\underbrace{0,\ldots,0}_{j}) \otimes V^{\mathfrak{sp}(1)_2}(j)\right) \otimes \left(\bigoplus_{k=0}^{m_1} V^{\mathfrak{sp}(m_1)}(\underbrace{1,\ldots,1}_{m_1-k},\underbrace{0,\ldots,0}_{k}) \otimes V^{\mathfrak{sp}(1)_1}(k)\right).
\end{aligned}$$

Note that the numbers

$$(1 \times (m_2 - j) + 0 \times j + [j]) + (1 \times (m_1 - k) + 0 \times k + [k]) = m_1 + m_2 = m.$$

3.4 $r = 5$

Recall that $\mathfrak{spin}(5) \cong \mathfrak{sp}(2)$ and

$$N_{\mathfrak{so}(8m)}(\mathfrak{spin}(5)) = \mathfrak{sp}(m) \oplus \mathfrak{sp}(2).$$

In this case

$$\mathbb{R}^{4m} \otimes \mathbb{C} \cong E \otimes H$$

where $E = \mathbb{C}^{2m}$ and $H = \mathbb{C}^4$ are the standard representations of $\mathfrak{sp}(m)$ and $\mathfrak{sp}(2)$ respectively.

In terms of the complexified standard representation $\mathbb{R}^{8m} \otimes \mathbb{C}$ of $SO(8m)$, its standard maximal torus looks like

$$\begin{pmatrix} e^{i\eta_1} & & & & & & \\ & e^{-i\eta_1} & & & & & \\ & & e^{i\eta_2} & & & & \\ & & & e^{-i\eta_2} & & & \\ & & & & \ddots & & \\ & & & & & e^{i\eta_{4m}} & \\ & & & & & & e^{-i\eta_{4m}} \end{pmatrix}$$

The standard maximal torus of $Sp(m)$ looks

$$\begin{pmatrix} e^{i\theta_1} & & & & \\ & e^{-i\theta_1} & & & \\ & & \ddots & & \\ & & & e^{i\theta_m} & \\ & & & & e^{-i\theta_m} \end{pmatrix}$$

and the standard maximal torus of $Sp(2)$ looks like

$$\begin{pmatrix} e^{i\varphi_1} & & & \\ & e^{-i\varphi_1} & & \\ & & e^{i\varphi_2} & \\ & & & e^{-i\varphi_2} \end{pmatrix}.$$

Clearly, we must take the Kronecker product of these two matrices so that we have the correspondence

$$\begin{aligned} \eta_{2j-1} &= \theta_j + \varphi_1 \\ \eta_{2j} &= \theta_j - \varphi_1 \\ \eta_{2m+2j-1} &= \theta_j + \varphi_2 \\ \eta_{2m+2j} &= \theta_j - \varphi_2, \end{aligned}$$

for $j = 1, \ldots, m$. Now, we substitute this identities into the weights of the Spin representation

$$\begin{aligned} \pm\frac{\eta_1}{2} \pm \cdots \pm \frac{\eta_{4m}}{2} &= \left(\pm\frac{\eta_1}{2} \pm \cdots \pm \frac{\eta_{2m}}{2}\right) + \left(\pm\frac{\eta_{2m+1}}{2} \pm \cdots \pm \frac{\eta_{4m}}{2}\right) \\ &= \sum_{j\in I_1} \frac{\theta_j + \varphi_1}{2} - \sum_{j\in \bar{I}_1} \frac{\theta_j + \varphi_1}{2} + \sum_{j\in I_2} \frac{\theta_j - \varphi_1}{2} - \sum_{j\in \bar{I}_2} \frac{\theta_j - \varphi_1}{2} \\ &\quad + \sum_{j\in I_3} \frac{\theta_j + \varphi_2}{2} - \sum_{j\in \bar{I}_3} \frac{\theta_j + \varphi_2}{2} + \sum_{j\in I_4} \frac{\theta_j - \varphi_2}{2} - \sum_{j\in \bar{I}_4} \frac{\theta_j - \varphi_2}{2} \\ &= (|I_1| - |I_2|)\varphi_1 + (|I_3| - |I_4|)\varphi_2 + \left(\sum_{j\in I_1\cap I_2} \theta_j - \sum_{j\in \bar{I}_1\cap \bar{I}_2} \theta_j\right) + \left(\sum_{j\in I_3\cap I_4} \theta_j - \sum_{j\in \bar{I}_3\cap \bar{I}_4} \theta_j\right), \end{aligned}$$

where $I_1, I_2, I_3, I_4 \subseteq \{1, \dots, m\}$. From here we see that

$$\Delta_{8m} = \bigoplus_{e=0}^{m} \bigoplus_{f=0}^{e} V^{\mathfrak{sp}(m)}(\underbrace{2, \dots, 2}_{m-e}, \underbrace{1, \dots, 1}_{e-f}, \underbrace{0, \dots, 0}_{f}) \otimes V^{\mathfrak{sp}(2)}(e, f).$$

Note that the numbers

$$2 \times (m-e) + 1 \times (e-f) + 0 \times f + [e+f] = 2m,$$

i.e. the entries of the coordinate vectors of the weights must add to $2m$.

We have a linear map

$$\mathfrak{t}_{sp(m)} \oplus \mathfrak{t}_{sp(1)} \hookrightarrow \mathfrak{t}_{SO(4m)}$$

expressed as

$$\begin{pmatrix}
1&&&&&1& \\
1&&&&&-1& \\
1&&&&&&1 \\
1&&&&&&-1 \\
&1&&&&1& \\
&1&&&&-1& \\
&1&&&&&1 \\
&1&&&&&-1 \\
&&1&&&1& \\
&&1&&&-1& \\
&&1&&&&1 \\
&&1&&&&-1 \\
&&&1&&1& \\
&&&1&&-1& \\
&&&1&&&1 \\
&&&1&&&-1 \\
&&&&1&1& \\
&&&&1&-1& \\
&&&&1&&1 \\
&&&&1&&-1
\end{pmatrix}
\begin{pmatrix} x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ y[1] \\ y[2] \end{pmatrix}
=
\begin{pmatrix}
x[1]+y[1] \\ x[1]-y[1] \\ x[1]+y[2] \\ x[1]-y[2] \\
x[2]+y[1] \\ x[2]-y[1] \\ x[2]+y[2] \\ x[2]-y[2] \\
x[3]+y[1] \\ x[3]-y[1] \\ x[3]+y[2] \\ x[3]-y[2] \\
x[4]+y[1] \\ x[4]-y[1] \\ x[4]+y[2] \\ x[4]-y[2] \\
x[5]+y[1] \\ x[5]-y[1] \\ x[5]+y[2] \\ x[5]-y[2]
\end{pmatrix}$$

where we have set $m = 5$ for the sake of clarity.

The transpose is

$$\begin{pmatrix}
1&1&1&1&&&&&&&&&&&&&&&& \\
&&&&1&1&1&1&&&&&&&&&&&& \\
&&&&&&&&1&1&1&1&&&&&&&& \\
&&&&&&&&&&&&1&1&1&1&&&& \\
&&&&&&&&&&&&&&&&1&1&1&1 \\
1&-1&&&1&-1&&&1&-1&&&1&-1&&&1&-1&& \\
&&1&-1&&&1&-1&&&1&-1&&&1&-1&&&1&-1
\end{pmatrix}$$

3.5 $r = 6$

In this case $\mathfrak{spin}(6) \cong \mathfrak{su}(4)$ and

$$N_{\mathfrak{so}(8m)}(\mathfrak{spin}(6)) = \mathfrak{u}(m) \oplus \mathfrak{su}(4).$$

We have

$$\mathbb{R}^{8m} \cong \mathbb{R}^m \otimes \tilde{\Delta}_6$$

$$\begin{aligned} \mathbb{R}^{8m} \otimes \mathbb{C} &= E \otimes H + \overline{E \otimes H} \\ &= E \otimes H + \overline{E} \otimes \overline{H} \end{aligned}$$

where $E = \mathbb{C}^m$ and $H = \mathbb{C}^4$ are the standard representations of $U(m)$ and $SU(4)$ respectively.

In terms of the complexified standard representation $\mathbb{R}^{8m} \otimes \mathbb{C}$ of $SO(8m)$, its standard maximal torus looks like

$$\begin{pmatrix} e^{i\eta_1} & & & & & & \\ & e^{-i\eta_1} & & & & & \\ & & e^{i\eta_2} & & & & \\ & & & e^{-i\eta_2} & & & \\ & & & & \ddots & & \\ & & & & & e^{i\eta_{4m}} & \\ & & & & & & e^{-i\eta_{4m}} \end{pmatrix}$$

The standard maximal torus of $U(m)$ looks

$$\begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_m} \end{pmatrix},$$

and the standard maximal torus of $SU(4)$ looks like

$$\begin{pmatrix} e^{i\varphi_3} & & & \\ & e^{i\varphi_2} & & \\ & & e^{i\varphi_1} & \\ & & & e^{-i(\varphi_1+\varphi_2+\varphi_3)} \end{pmatrix}.$$

Clearly we must take Kronecker product of this matrices, and we can identify

$$\begin{aligned} \eta_{4j-3} &= \theta_j + \varphi_3, \\ \eta_{4j-2} &= \theta_j + \varphi_2, \\ \eta_{4j-1} &= \theta_j + \varphi_1, \\ \eta_{4j} &= \theta_j - \varphi_1 - \varphi_2 - \varphi_3, \end{aligned}$$

for $j = 1, \ldots, m$.

On the other hand, the representation $\mathbb{C}^m$ of $U(m)$ factors as

$$\mathbb{C}^m \cong L \otimes E''$$

with respect to the isomorphism

$$U(m) \cong U(1) \times_{\mathbb{Z}_m} SU(m),$$

where L and E'' denote the fundamental 1-dimensional and m-dimensional representations of $U(1)$ and $SU(m)$ respectively. The standard maximal torus of $U(m)$ factors accordingly as

$$\begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_m} \end{pmatrix} = e^{i\theta'} \begin{pmatrix} e^{i\theta''_1} & & & \\ & e^{i\theta''_2} & & \\ & & \ddots & \\ & & & e^{-i(\theta''_1+\cdots+\theta''_{m-1})} \end{pmatrix}$$

so that

$$\theta_j = \theta' + \theta''_j$$

for $j = 1, \ldots, m-1$, and

$$\theta_m = \theta' - (\theta''_1 + \cdots + \theta''_{m-1})$$

Thus,

$$\begin{aligned} \eta_{4j-3} &= \theta' + \theta''_j + \varphi_3, \\ \eta_{4j-2} &= \theta' + \theta''_j + \varphi_2, \\ \eta_{4j-1} &= \theta' + \theta''_j + \varphi_1, \\ \eta_{4j} &= \theta' + \theta''_j - \varphi_1 - \varphi_2 - \varphi_3, \end{aligned}$$

for $j = 1, \ldots, m-1$, and

$$\begin{aligned} \eta_{4m-3} &= \theta' - \theta''_1 - \ldots - \theta''_{m-1} + \varphi_3, \\ \eta_{4m-2} &= \theta' - \theta''_1 - \ldots - \theta''_{m-1} + \varphi_2, \\ \eta_{4m-1} &= \theta' - \theta''_1 - \ldots - \theta''_{m-1} + \varphi_1, \\ \eta_{4m} &= \theta' - \theta''_1 - \ldots - \theta''_{m-1} - \varphi_1 - \varphi_2 - \varphi_3. \end{aligned}$$

Note that each weight

$$\begin{aligned} \pm\frac{\eta_1}{2} \pm \cdots \pm \frac{\eta_{4m}}{2} &= \sum_{a=1}^{3}\sum_{j=1}^{m-1}\left(\pm\frac{\eta_{4j-a}}{2}\right) + \sum_{j=1}^{m-1}\left(\pm\frac{\eta_{4j}}{2}\right) \pm \frac{\eta_{4m-3}}{2} \pm \frac{\eta_{4m-2}}{2} \pm \frac{\eta_{4m-1}}{2} \pm \frac{\eta_{4m}}{2} \\ &= \sum_{a=1}^{3}\left(\sum_{j=1}^{m-1} \pm\frac{\theta' + \theta''_j + \varphi_a}{2}\right) \\ &\quad + \sum_{j=1}^{m-1}\left(\pm\frac{\theta' + \theta''_j - \varphi_1 - \varphi_2 - \varphi_3}{2}\right) \\ &\quad \pm\frac{\theta' - \theta''_1 - \ldots - \theta''_{m-1} + \varphi_3}{2} \\ &\quad \pm\frac{\theta' - \theta''_1 - \ldots - \theta''_{m-1} + \varphi_2}{2} \\ &\quad \pm\frac{\theta' - \theta''_1 - \ldots - \theta''_{m-1} + \varphi_1}{2} \\ &\quad \pm\frac{\theta' - \theta''_1 - \ldots - \theta''_{m-1} - \varphi_1 - \varphi_2 - \varphi_3}{2} \\ &= \sum_{a=1}^{3}\left(\sum_{j\in I_a}\frac{\theta' + \theta''_j + \varphi_a}{2} - \sum_{j\in \bar{I}_a}\frac{\theta' + \theta''_j + \varphi_a}{2}\right) \end{aligned}$$

$$+ \left(\sum_{j \in I_0} \frac{\theta' + \theta''_j - \varphi_1 - \varphi_2 - \varphi_3}{2} - \sum_{j \in \bar{I}_0} \frac{\theta' + \theta''_j - \varphi_1 - \varphi_2 - \varphi_3}{2} \right)$$

$$\pm \frac{\theta' - \theta''_1 - \ldots - \theta''_{m-1} + \varphi_3}{2}$$

$$\pm \frac{\theta' - \theta''_1 - \ldots - \theta''_{m-1} + \varphi_2}{2}$$

$$\pm \frac{\theta' - \theta''_1 - \ldots - \theta''_{m-1} + \varphi_1}{2}$$

$$\pm \frac{\theta' - \theta''_1 - \ldots - \theta''_{m-1} - \varphi_1 - \varphi_2 - \varphi_3}{2}$$

3.6 $r = 7$

In this case

$$N_{\mathfrak{so}(8m)}(\mathfrak{spin}(7)) = \mathfrak{so}(m) \oplus \mathfrak{spin}(7).$$

We have

$$\mathbb{R}^{8m} = \mathbb{R}^m \otimes \tilde{\Delta}_7$$

and

$$\mathbb{R}^{8m} \otimes \mathbb{C} = E \otimes \Delta_7$$

where E and Δ_7 are the standard complex representations of $\mathfrak{so}(m)$ and $\mathfrak{spin}(7)$.

In terms of the complexified standard representation $\mathbb{R}^{8m} \otimes \mathbb{C}$ of $SO(8m)$, its standard maximal torus looks like

$$\begin{pmatrix} e^{i\eta_1} & & & & & & \\ & e^{-i\eta_1} & & & & & \\ & & e^{i\eta_2} & & & & \\ & & & e^{-i\eta_2} & & & \\ & & & & \ddots & & \\ & & & & & e^{i\eta_{4m}} & \\ & & & & & & e^{-i\eta_{4m}} \end{pmatrix}$$

The standard maximal torus of $SO(m)$ looks

$$\begin{pmatrix} e^{i\theta_1} & & & & \\ & e^{-i\theta_1} & & & \\ & & \ddots & & \\ & & & e^{i\theta_{m/2}} & \\ & & & & e^{-i\theta_{m/2}} \end{pmatrix}$$

if m is even, and

$$\begin{pmatrix} e^{i\theta_1} & & & & & \\ & e^{-i\theta_1} & & & & \\ & & \ddots & & & \\ & & & e^{i\theta_{(m-1)/2}} & & \\ & & & & e^{-i\theta_{(m-1)/2}} & \\ & & & & & 1 \end{pmatrix}$$

if m is odd. From the description in Section 2.2, the standard maximal torus of $Spin(7)$

$$(\cos(\varphi_1/2) + \sin(\varphi_1/2)e_1e_2)(\cos(\varphi_2/2) + \sin(\varphi_2/2)e_1e_2)(\cos(\varphi_3/2) + \sin(\varphi_3/2)e_1e_2)$$

looks like

$$\begin{pmatrix} e^{\frac{i(\varphi_1+\varphi_2+\varphi_3)}{2}} & & & & & & & \\ & e^{\frac{i(-\varphi_1+\varphi_2+\varphi_3)}{2}} & & & & & & \\ & & e^{\frac{i(\varphi_1-\varphi_2+\varphi_3)}{2}} & & & & & \\ & & & e^{\frac{i(-\varphi_1-\varphi_2+\varphi_3)}{2}} & & & & \\ & & & & e^{\frac{i(\varphi_1+\varphi_2-\varphi_3)}{2}} & & & \\ & & & & & e^{\frac{i(-\varphi_1+\varphi_2-\varphi_3)}{2}} & & \\ & & & & & & e^{\frac{i(\varphi_1-\varphi_2-\varphi_3)}{2}} & \\ & & & & & & & e^{\frac{i(-\varphi_1-\varphi_2-\varphi_3)}{2}} \end{pmatrix}$$

3.6.1 m even

Let us suppose $m = 2p$. By taking the Kronecker product of these matrices we can identify

$$\begin{aligned}
\eta_{8j-7} &= \theta_j + \frac{\varphi_1 + \varphi_2 + \varphi_3}{2}, \\
\eta_{8j-6} &= \theta_j + \frac{-\varphi_1 + \varphi_2 + \varphi_3}{2}, \\
\eta_{8j-5} &= \theta_j + \frac{\varphi_1 - \varphi_2 + \varphi_3}{2}, \\
\eta_{8j-4} &= \theta_j + \frac{-\varphi_1 - \varphi_2 + \varphi_3}{2}, \\
\eta_{8j-3} &= \theta_j + \frac{\varphi_1 + \varphi_2 - \varphi_3}{2}, \\
\eta_{8j-2} &= \theta_j + \frac{-\varphi_1 + \varphi_2 - \varphi_3}{2}, \\
\eta_{8j-1} &= \theta_j + \frac{\varphi_1 - \varphi_2 - \varphi_3}{2}, \\
\eta_{8j} &= \theta_j + \frac{-\varphi_1 - \varphi_2 - \varphi_3}{2},
\end{aligned}$$

for $j = 1, \ldots, p$. Thus,

$$\begin{aligned}
\sum_{j=1}^{4m} \left(\pm \frac{\eta_j}{2}\right) &= \sum_{a=0}^{7} \sum_{j=1}^{m} \left(\pm \frac{\eta_{8j-a}}{2}\right) \\
&= \sum_{a=0}^{7} \left(\sum_{j \in I_a} \frac{\eta_{8j-a}}{2} - \sum_{j \in \bar{I}_a} \frac{\eta_{8j-a}}{2} \right) \\
&= \sum_{j \in I_0} \frac{\eta_{8j}}{2} - \sum_{j \in \bar{I}_0} \frac{\eta_{8j}}{2} \\
&\quad + \sum_{j \in I_1} \frac{\eta_{8j-1}}{2} - \sum_{j \in \bar{I}_1} \frac{\eta_{8j-1}}{2} \\
&\quad + \sum_{j \in I_2} \frac{\eta_{8j-2}}{2} - \sum_{j \in \bar{I}_2} \frac{\eta_{8j-2}}{2} \\
&\quad + \sum_{j \in I_3} \frac{\eta_{8j-3}}{2} - \sum_{j \in \bar{I}_3} \frac{\eta_{8j-3}}{2} \\
&\quad + \sum_{j \in I_4} \frac{\eta_{8j-4}}{2} - \sum_{j \in \bar{I}_4} \frac{\eta_{8j-4}}{2} \\
&\quad + \sum_{j \in I_5} \frac{\eta_{8j-5}}{2} - \sum_{j \in \bar{I}_5} \frac{\eta_{8j-5}}{2} \\
&\quad + \sum_{j \in I_6} \frac{\eta_{8j-6}}{2} - \sum_{j \in \bar{I}_6} \frac{\eta_{8j-6}}{2} \\
&\quad + \sum_{j \in I_7} \frac{\eta_{8j-7}}{2} - \sum_{j \in \bar{I}_7} \frac{\eta_{8j-7}}{2}.
\end{aligned}$$

3.6.2 m odd

Now, let us suppose $m = 2p + 1$. By taking the Kronecker product of these matrices we can identify

$$\begin{aligned}
\eta_{8j-7} &= \theta_j + \frac{\varphi_1 + \varphi_2 + \varphi_3}{2}, \\
\eta_{8j-6} &= \theta_j + \frac{-\varphi_1 + \varphi_2 + \varphi_3}{2}, \\
\eta_{8j-5} &= \theta_j + \frac{\varphi_1 - \varphi_2 + \varphi_3}{2}, \\
\eta_{8j-4} &= \theta_j + \frac{-\varphi_1 - \varphi_2 + \varphi_3}{2}, \\
\eta_{8j-3} &= \theta_j + \frac{\varphi_1 + \varphi_2 - \varphi_3}{2}, \\
\eta_{8j-2} &= \theta_j + \frac{-\varphi_1 + \varphi_2 - \varphi_3}{2}, \\
\eta_{8j-1} &= \theta_j + \frac{\varphi_1 - \varphi_2 - \varphi_3}{2}, \\
\eta_{8j} &= \theta_j + \frac{-\varphi_1 - \varphi_2 - \varphi_3}{2},
\end{aligned}$$

for $j = 1, \ldots, p$, and

$$\begin{aligned}
\eta_{4m-3} &= \frac{\varphi_1 + \varphi_2 + \varphi_3}{2}, \\
\eta_{4m-2} &= \frac{-\varphi_1 + \varphi_2 + \varphi_3}{2}, \\
\eta_{4m-1} &= \frac{\varphi_1 - \varphi_2 + \varphi_3}{2}, \\
\eta_{4m} &= \frac{-\varphi_1 - \varphi_2 + \varphi_3}{2}.
\end{aligned}$$

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